Notes on ratios of source spectra

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These notes are concerned with the function describing the ratio of two Fourier spectra, as might be computed in empirical Green's function analyzes of a mainshock and one of its aftershocks, the goal being to determine the stress parameter of the mainshock (and secondarily, the aftershock) without making assumptions about geometrical spreading or site response.

Consider the ratio of two single-corner frequency source spectral models, with displacement spectra going as $\omega^{-\gamma}$ and ω^{-p}:

$$S_1 = M_{01} \Big/ \left[1 + \left(f / f_{c1}\right)^{\xi}\right]^{\gamma/\xi} \tag{1}$$

$$S_2 = M_{02} \Big/ \left[1 + \left(f / f_{c2}\right)^{\eta}\right]^{p/\eta} \tag{2}$$

The form of these spectral models is a slight generalization of the form used by Chael and Kromer (1988). This form covers most single-corner-frequency models of the source spectrum. Considering just equation (1) for illustration, the most commonly used the ω^{-2} source model is given by choosing $\gamma = \xi = 2$, whereas choosing $\gamma = 2$ and $\xi = 4$ leads to an ω^{-2} source model that has a sharper bend at the corner frequency, being down by a factor of $1/\sqrt{2}$ rather than $1/2$ at the corner frequency.

The ratio of the spectra given by equations (1) and (2) is

$$\frac{S_1}{S_2} = \left(\frac{M_{01}}{M_{02}}\right) \frac{\left[1 + \left(f / f_{c2}\right)^{\eta}\right]^{p/\eta}}{\left[1 + \left(f / f_{c1}\right)^{\xi}\right]^{\gamma/\xi}} \tag{3}$$

For fitting observed ratios, it is convenient to recast equation (3) in terms of the low-frequency level LFL, the corner frequency f_{c1} of the numerator (presumed larger) event, and the value of the ratio (HFL) at $f = f_{hf}$, where f_{hf} is high enough that the second terms in equation (3) are much greater than unity. Then

$$LFL = \frac{M_{01}}{M_{02}} \tag{4}$$

and

$$HFL = \left(\frac{M_{01}}{M_{02}}\right)\left(f_{c1}/f_{c2}\right)^{p}\left(f_{hf}/f_{c1}\right)^{p-\gamma}. \tag{5}$$

With the relations in equations (3), (4), and (5), the ratio of spectra for two events can be written in the following form, which is convenient when fitting a curve manually to observed ratios:

$$\frac{S_1}{S_2} = LFL\frac{\left[1+\left(HFL/LFL\right)^{\eta/p}\left(f_{hf}/f_{c1}\right)^{\eta\left(\frac{\gamma}{p}-1\right)}\left(f/f_{c1}\right)^{\eta}\right]^{p/\eta}}{\left[1+\left(f/f_{c1}\right)^{\xi}\right]^{\gamma/\xi}}. \tag{6}$$

This is convenient because the four parameters, LFL, f_{c1}, HFL, and f_{hf}, can be estimated visually from the observed ratios, whereas it is more difficult to estimate f_{c2}.

If $\gamma = p$, (both spectra have the same high-frequency decay), equation (6) simplifies to:

$$\frac{S_1}{S_2} = LFL\frac{\left[1+\left(HFL/LFL\right)^{\eta/\gamma}\left(f/f_{c1}\right)^{\eta}\right]^{\gamma/\eta}}{\left[1+\left(f/f_{c1}\right)^{\xi}\right]^{\gamma/\xi}}, \tag{7}$$

and equation (5) becomes:

$$HFL = \left(\frac{M_{01}}{M_{02}}\right)\left(f_{c1}/f_{c2}\right)^{p}. \tag{8}$$

(the high-frequency asymptote is flat, and therefore a high-frequency reference frequency f_{hf} need not be specified).

From equations (4) and (8), the second corner frequency is given by:

$$f_{c2} = f_{c1} \left(\frac{LFL}{HFL} \right)^{1/p} \quad (9)$$

(equation (5) can be solved for f_{c2} in the general case).

Equation (7) is further simplified if $\gamma = \xi = p = \eta = 2$ (the common ω^{-2} model; these values are used in the rest of this note):

$$\frac{S_1}{S_2} = LFL \frac{\left[1 + \left(HFL/LFL \right) \left(f / f_{c1} \right)^2 \right]}{\left[1 + \left(f / f_{c1} \right)^2 \right]}. \quad (10)$$

Given LFL and the equation defining the moment magnitude (for which $\log M_0 \sim 1.5\mathbf{M}$), the difference of moment magnitudes is given by

$$\mathbf{M}_1 - \mathbf{M}_2 = \frac{2}{3} \log \left(LFL \right). \quad (11)$$

This equation is independent of the values of γ, ξ, *p*, and η.

So far no assumption has been made about the stress parameters of the two events. Using the standard relation

$$M_0 f_c^3 \sim \Delta\sigma , \quad (12)$$

Equations (4) and (9) give

$$\frac{\Delta\sigma_1}{\Delta\sigma_2} = HFL \left(\frac{HFL}{LFL} \right)^{1/p}. \quad (13)$$

Equality of the stress parameters requires

$$HFL = LFL^{1/(1+p)}. \quad (14)$$

or

$$LFL = HFL^{1+p} \quad (15)$$

The stress parameter is given as a function of the seismic moment M_0, the source corner frequency f_c, and the shear-wave velocity in the vicinity of the source β_s by the relation

$$\Delta\sigma = 8.47 M_0 \left(\frac{f_c}{\beta_s}\right)^3 \tag{16}$$

where all quantities are in the same system of units; in terms of the usual mixed units (stress in bars, moment in dyne-cm, shear-wave velocity in km/s), this becomes

$$\Delta\sigma = 8.47 \times 10^{-21} M_0 \left(\frac{f_c}{\beta_s}\right)^3 . \tag{17}$$

(As a side note, brought up by Ralph Archuleta's comments during the 11-13 October 2011 PEER NGA-East workshop, the relation used by Allmann and Shearer (2009) is

$$\Delta\sigma = 13.5 M_0 \left(\frac{f_c}{\beta_s}\right)^3 . \tag{18}$$

The stress parameters derived for the same moment and corner frequency will differ by a factor of 1.6.)

References

Allmann, B. P. and P. M. Shearer (2009). Global variations of stress drop for moderate to large earthquakes, *J. Geophys. Res.* **114**, B01310, doi:10.1029/2008JB005821, 22 pp.

Chael, E. P. and Kromer, R. P. (1988). High-frequency spectral scaling of a main shock/aftershock sequence near the Norwegian coast, *Bull. Seismol. Soc. Am.* **78**, 561-570.